JOURNAL OF COMPUTATIONAL PHYSICS **136,** 100–114 (1997)
ARTICLE NO. CP975747

A Spectral Model for Two-Dimensional Incompressible Fluid Flow in a Circular Basin

I. Mathematical Formulation

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Received August 12, 1996; revised May 14, 1997

A spectral numerical scheme is developed for simulations of two-dimensional incompressible fluid flow in a circular basin. The vorticity and streamfunction fields are represented by products of Jacobi polynomials and complex exponentials. The Jacobi polynomials are used for the radial dependence of the fields, and the complex exponentials for the angular dependence. The basis functions are orthogonal with respect to the natural inner product for a circular domain. It is demonstrated how the Laplace operator and its inverse can be expressed exactly in terms of these basis functions. It is also shown how the advection term can be evaluated without aliasing, making use of a transform grid with equidistant angular values and Gaussian radial values. It is shown that without forcing and friction the model conserves absolute enstrophy and circulation and, if the planetary vorticity is circularly symmetric, also angular momentum. The model does not conserve energy. However, the degree of conservation of energy rapidly increases with increasing resolution. Examples of time integrations will be discussed in the companion paper (Part II; W. T. M. Verkley, 1997, *J. Comput. Phys.* 115–131 **136**).

1. INTRODUCTION

The study of two-dimensional incompressible fluid flow is of considerable interest. In many cases the flow of three-dimensional fluid systems is organized in such a way that its behavior is essentially two-dimensional. Examples are plasma systems in a magnetic field in which the magnetic field forces the motion to be two-dimensional [6]. Other examples are large-scale atmospheric and oceanic flows in which vertical density stratification and the rotation of the earth force the flow to behave layerwise two-dimensionally [9]. Understanding the dynamics of two-dimensional fluid flows is therefore crucial in grasping the dynamics of important three-dimensional systems, such as magnetized plasmas and the earth's atmosphere and oceans.

The present paper (Part I) and its companion (Part II) [14] have their origin in the study by Verkley and Zimmerman [12], hereafter referred to as VZ, on a simple model of the nonlinear wind-driven ocean circulation. The model describes an ocean in hydrostatic and geostrophic balance with water of constant density, bounded by rigid upper and lower boundaries. It can be shown [9] that this system is governed by the horizontal advection of absolute vorticity q with a nondivergent velocity field $\mathbf{v} = \mathbf{k} \times \nabla\psi$, where $\mathbf{k}$ is a unit vector pointing vertically upward and ψ is a streamfunction. The basin was taken to be circular, with ψ zero at the boundary (free-slip), and the flow was assumed to be forced by a temporally constant and spatially uniform input of vorticity and damped by Ekman friction. The equation studied was the time-independent version of

$$\frac{\partial q}{\partial t} + J(\psi, q) + \tau\phi + \kappa\zeta = 0. \tag{1}$$

Here q is the absolute vorticity, $q = \zeta + f$, where the relative vorticity $\zeta = \mathbf{k} \cdot \nabla \times \mathbf{v} = \nabla^2\psi$ and f is the planetary vorticity. The operators ∇^2 and J are the Laplace and Jacobi operators, respectively. Use of the latter operator is common practice in writing the advection term $\mathbf{v} \cdot \nabla q$ in terms of a streamfunction. The forcing is $-\tau\phi$ and the Ekman friction is $-\kappa\zeta$, where ϕ is the spatial structure of the forcing and τ and κ are numbers measuring the strength of the forcing and the friction. We will also consider the case in which the friction is given by a viscosity term $\nu\nabla^2\zeta$ on the right-hand side of (1). In this case we impose as an extra boundary condition that the radial derivative of ψ equals the velocity of the boundary (no-slip). We note that, if the forcing and Ekman friction are zero, the system describes an inviscid two-dimensional and incompressible fluid.

In the study of VZ steady states of (1) were obtained by expanding this equation in a perturbation series in the inverse Ekman number κ^{-1}. For a limited part of the parameter space the series converged and steady states could be found. The present model is aimed at checking whether these steady states can be reproduced by long integrations of a time-dependent model. In the same way we hope to

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obtain steady solutions in the region of parameter space where the series does not converge. Furthermore, we expect the model to be useful in its own right as a research tool in the study of two-dimensional flows in simple bounded basins. The circular form of the basin facilitates, e.g., the application of statistical mechanical techniques to fluid systems, the results of which might again be checked by long time integrations with the proposed model.

In Section 2 the basis functions are introduced in terms of which the spatial structure of the streamfunction and absolute vorticity fields are discretized. It is explained how these functions can be used to represent the different operators in (1). The section continues with the formulation of the time-evolution model and a discussion of its conserved quantities. The section ends with the description of a procedure to incorporate the no-slip boundary condition in the case of flow with viscosity. A more detailed exposition of the properties of the basis functions, in particular of the evaluation of the Laplace operator and its inverse, can be found in Appendices A and B. Section 3 closes the paper with a summary.

2. THE SPECTRAL METHOD

In this section we will explain how the spectral model is constructed. The model in its basic form only respects the boundary condition that the streamfunction ψ is zero at the boundary, implying no normal flow at the boundary (free-slip). The vorticity field ζ, on the other hand, is completely free. Extra boundary conditions, like the condition that $\partial\psi/\partial r$ must be equal to a given velocity distribution at the boundary (no-slip), impose constraints on the vorticity field. How these constraints can be incorporated in the model will be discussed at the end of this section.

2.1. *Basis Functions of the Model*

The basic geometric parameters of the system are given in Fig. 1. The model basin has a circular boundary with radius 1, assuming that lengths are expressed in units of the actual radius R of the basin. Points within the basin are denoted by the rectangular coordinates x and y, or by the polar coordinates r and θ. The basis functions that will be used to discretize the spatial structures of the fields in (1) are called $Y_{mn}(r, \theta)$, where m is an integer running from $-\infty$ to $+\infty$ and n is an integer assuming the values $|m|, |m| + 2, |m| + 4$, ..., up to ∞. The functions Y_{mn} are defined by

$$Y_{mn}(r, \theta) \equiv W_{mn}(r)e^{im\theta}, \tag{2}$$

where

$$W_{mn}(r) \equiv r^{|m|}P_k^{(0,|m|)}(s) \tag{3}$$

and $P_k^{(\alpha,\beta)}(s)$ is a Jacobi polynomial with argument $s = 2r^2 - 1$ and degree $2k$ with $k = (n - |m|)/2$ (see Abramowitz and Stegun [1, Chap. 22]). The basis functions Y_{mn}

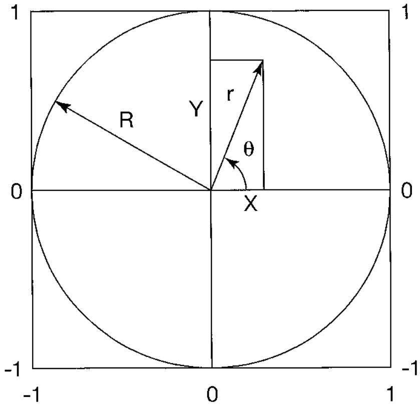

FIG. 1. The basic geometry of the system. The flow domain is bounded by a circle with radius R. On this circle the streamfunction is assumed to be zero, implying no normal flow at the boundary. The coordinates used are the rectangular coordinates x and y and the polar coordinates r and θ. Lengths are measured in units of R.

span the same linear function space as the functions X_{mn}, defined by

$$X_{mn}(r, \theta) \equiv V_{mn}(r)e^{im\theta}, \tag{4}$$

where

$$V_{mn}(r) \equiv r^{|m|}r^{2k} = r^n. \tag{5}$$

It can be verified that the functions X_{mn}, in turn, span the same space as the functions 1, x, y, x^2, xy, y^2, x^3, x^2y, xy^2, y^3, etc. As the latter functions are regular (infinitely differentiable) at the origin, the index n in (2) and (4) must start with $|m|$ and increase in steps of 2. Because polynomials in x and y form a complete set in terms of which any square integrable function of x and y on a bounded domain can be represented, the sets of functions X_{mn} and Y_{mn} are complete too. Note that for both Y_{mn} and X_{mn} the radial dependence is a polynomial of degree n. The transformation formulas between the functions Y_{mn} and X_{mn} are given in Appendix A.

For different values of m both the functions Y_{mn} and X_{mn} are orthogonal with respect to the following inner product,

$$\langle \xi_1, \xi_2 \rangle \equiv \frac{1}{\pi}\int_0^1\int_0^{2\pi} r\,dr\,d\theta\,\xi_1^*\xi_2, \tag{6}$$

where the asterisk denotes complex conjugation. The difference between Y_{mn} and X_{mn} is that the former are also orthogonal for different values of n with respect to the same inner product. More explicitly, we have

$$\langle Y_{m'n'}, Y_{mn} \rangle = \frac{\delta_{m'm}\delta_{n'n}}{n+1}, \tag{7}$$

where δ is the Kronecker delta. As the functions Y_{mn} form a complete set in terms of which any square integrable field $\xi(r, \theta)$ can be expanded, we have

$$\xi = \sum_{m=-\infty}^{+\infty} \sum_{n=|m|}^{\infty} \xi_{mn} Y_{mn}, \tag{8}$$

where, due to the orthogonality condition (7), we have for the expansion coefficients

$$\xi_{mn} = (n+1)\langle Y_{mn}, \xi \rangle. \tag{9}$$

It is understood that in (8) and in summations of the same kind the index n assumes the values $|m|$, $|m| + 2$, $|m| + 4$, etc. More information and proofs of the different conjectures can be found in Appendix A.

We now introduce two finite-dimensional subspaces, called TN and UN. The space TN is spanned by the functions X_{mn} or Y_{mn} with m running from $-N$ to $+N$ and n running, in steps of 2, from $|m|$ to maximally N. The space UN is spanned by the functions X_{mn} or Y_{mn} with m running from $-N$ to $+N$, but n running from $|m|$ to maximally $N+2$. As a result of the orthogonality of the basis functions Y_{mn} we can associate two projection operators, $\mathscr{P}_N$ and $\mathscr{Q}_N$, that project on these subspaces,

$$\mathscr{P}_N \xi \equiv \sum_{m=-N}^{+N} \sum_{n=|m|}^{N} \xi_{mn} Y_{mn}, \tag{10}$$

$$\mathscr{Q}_N \xi \equiv \sum_{m=-N}^{+N} \sum_{n=|m|}^{N+2} \xi_{mn} Y_{mn}, \tag{11}$$

where ξ_{mn} are the expansion coefficients (9). In the spectral model to be developed we assume that all fields with the dimension of vorticity—such as q, ζ, f, and ϕ, collectively denoted by η—are elements of the space TN. Later in this section it will be shown that fields with the dimension of the streamfunction—such as ψ, collectively denoted by χ—are then elements of UN. If we mark the finite-dimensional representation of vorticity fields by means of a hat and the finite-dimensional representation of streamfunction fields by means of a tilde, we thus write

$$\hat{\eta} = \sum_{m=-N}^{+N} \sum_{n=|m|}^{N} \eta_{mn} Y_{mn}, \tag{12}$$

$$\tilde{\chi} = \sum_{m=-N}^{+N} \sum_{n=|m|}^{N+2} \chi_{mn} Y_{mn}. \tag{13}$$

Note that if the fields $\hat{\eta}$ and $\tilde{\chi}$ are real, which will be the case in practice, we should have that $\eta_{-mn} = \eta^*_{mn}$ and $\chi_{-mn} = \chi^*_{mn}$. We therefore only need to keep track of the real and imaginary values of the coefficients for nonnegative values of m. A few examples of the functions W_{mn} that appear in the definition of Y_{mn} are given in Table I. We represent the latter functions by a diagram in which each function is denoted by a dot in a lattice of which the coordinates are m and n, as illustrated by Fig. 2. The T in the name of the finite-dimensional subspace TN refers to the triangular shape of the lattice of points representing the space TN in Fig. 2, in analogy to the usage in spectral models for a sphere based on spherical harmonics.

2.2. *Representation of the Operators*

The operators that concern the spatial structure of the fields are the Laplacian ∇^2 and the Jacobian J, for which we have

$$\nabla^2 \chi = \frac{\partial^2 \chi}{\partial x^2} + \frac{\partial^2 \chi}{\partial y^2} = \frac{1}{r}\frac{\partial}{\partial r}\left(r \frac{\partial \chi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 \chi}{\partial \theta^2}, \tag{14}$$

$$J(\chi, \eta) = \frac{\partial \chi}{\partial x}\frac{\partial \eta}{\partial y} - \frac{\partial \chi}{\partial y}\frac{\partial \eta}{\partial x} = \frac{1}{r}\left(\frac{\partial \chi}{\partial r}\frac{\partial \eta}{\partial \theta} - \frac{\partial \chi}{\partial \theta}\frac{\partial \eta}{\partial r}\right). \tag{15}$$

We first discuss the representation of the Laplace operator ∇^2 and its inverse ∇^{-2} in terms of the basis functions Y_{mn}. The Laplacian and its inverse of the functions X_{mn} can readily be found and expressed in the same functions; see (A22) and (A33) of Appendix A. In expression (A33)

TABLE I

A Few Examples of the Functions $W_{mn}(r)$ as Defined in (3)

7		$r(35r^6 - 60r^4 + 30r^2 - 4)$		$r^3(21r^4 - 30r^2 + 10)$		$r^5(7r^2 - 6)$
6	$20r^6 - 30r^4 + 12r^2 - 1$		$r^2(15r^4 - 20r^2 + 6)$		$r^4(6r^2 - 5)$	
5		$r(10r^4 - 12r^2 + 3)$		$r^3(5r^2 - 4)$		r^5
4	$6r^4 - 6r^2 + 1$		$r^2(4r^2 - 3)$		r^4	
3		$r(3r^2 - 2)$		r^3		
2	$2r^2 - 1$		r^2			
1		r				
0	1					
	0	1	2	3	4	5

Note. The columns correspond to different values of m and the rows to different values of n. The table contains all the functions of a $T5$ spectral model.

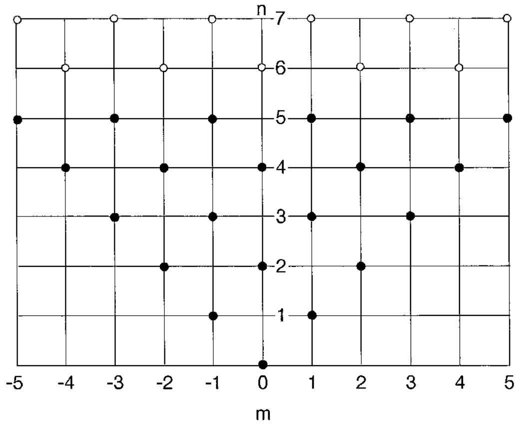

FIG. 2. Graphical representation of the functions $Y_{mn}(r,\ \theta) \equiv W_{mn}(r)e^{im\theta}$. Each dot, with coordinates m and n, is to be associated with a function Y_{mn}. Note that the n values increase in steps of 2. The solid dots shown in this figure represent the space $T5$. The open dots are the extra functions in the space $U5$ needed to represent the streamfunction associated with any vorticity field in $T5$. The functions $W_{mn}(r)$ corresponding to all the dots in this figure can be found in Table I.

the functions $X_{m|m|}$ are used to satisfy the boundary condition that the inverse Laplacian of X_{mn} is zero at $r = 1$. This is possible because the Laplacian of the functions $X_{m|m|}$ is zero, as can be verified readily from (4), (5), and (14). Using the transformation formulas between X_{mn} and Y_{mn}, it is shown in Appendix A how the Laplacian and its inverse can be found for Y_{mn} and expressed in the same functions. The result is

where $[\cdots]_{mn}$ denotes the expansion coefficients of the field between the square brackets.

We next discuss the method by means of which the Jacobian in (1) can be calculated. We recall that any streamfunction $\tilde{\chi}$ belongs to UN and that any vorticity $\hat{\eta}$ belongs to TN. This means that $\tilde{\chi}$ is a polynomial in r with maximum degree $N + 2$ and a trigonometric series in θ with maximum wavenumber N. In the same way, the field $\hat{\eta}$ is a polynomial in r with maximum degree N and a trigonometric series in θ with maximum wavenumber N. From the definition of the Jacobian in (15) it then follows that $J(\tilde{\chi},\ \hat{\eta})$ is a polynomial in r with maximum degree $2N$ and a trigonometric series in θ with maximum wavenumber $2N$. So the Jacobian of $\tilde{\chi}$ and $\hat{\eta}$ can be written

$$J(\tilde{\chi},\ \hat{\eta}) = \sum_{m=-2N}^{2N} \sum_{n=|m|}^{2N} J_{mn} Y_{mn}, \tag{18}$$

where J_{mn} are the corresponding coefficients given by (see (8) and (9))

$$J_{mn} = \frac{n+1}{\pi} \int_0^1 \int_0^{2\pi} r\, dr\, d\theta Y^*_{mn} J(\tilde{\chi},\ \hat{\eta}). \tag{19}$$

In spectral models, like the one to be developed here, we project this Jacobian on the space TN. This means that we only need to calculate the coefficients J_{mn} for $-N \leq m \leq N$ and $|m| \leq n \leq N$. For these coefficients the integrand in (19) is a polynomial in r with maximum degree $3N$ and a trigonometric series in θ with wavenumbers not ex-

$$\nabla^2 Y_{mn} = \sum_{n'=|m|}^{n-2} (n' + 1)(n - n')(n + n' + 2) Y_{mn'}, \tag{16a}$$

$$\nabla^{-2} Y_{mn} = \begin{cases} -[4(n+1)(n+2)]^{-1} Y_{mn} + [4(n+1)(n+2)]^{-1} Y_{mn+2} & (n = |m|) \\ [4n(n+1)]^{-1} Y_{mn-2} - [2n(n+2)]^{-1} Y_{mn} + [4(n+1)(n+2)]^{-1} Y_{mn+2} & (n \geq |m| + 2). \end{cases} \tag{16b}$$

We see that if $\tilde{\chi}$ is an element of UN then $\nabla^2\tilde{\chi}$ is contained in TN. Furthermore, if $\hat{\eta}$ is an element of TN then $\nabla^{-2}\hat{\eta}$ is contained in UN. It can be deduced from expressions (16) that we have

ceeding $3N$. This means that the integral in (19) can be calculated exactly by summation if the number of points is large enough. In fact, the integral over θ can be replaced by a summation over K equidistant values $\theta_i^K = (i - 1)$

$$[\nabla^2 \tilde{\chi}]_{mn} = \sum_{n'=n+2}^{N+2} (n + 1)(n' - n)(n' + n + 2) \chi_{mn'}, \tag{17a}$$

$$[\nabla^{-2} \hat{\eta}]_{mn} = \begin{cases} -[4(n+1)(n+2)]^{-1} \eta_{mn} + [4(n+2)(n+3)]^{-1} \eta_{mn+2} & (n = |m|) \\ [4n(n-1)]^{-1} \eta_{mn-2} - [2n(n+2)]^{-1} \eta_{mn} + [4(n+2)(n+3)]^{-1} \eta_{mn+2} & (n \geq |m| + 2), \end{cases} \tag{17b}$$

TABLE II

Values of the Maximum Value $N_{\max}$ of the Truncation Limit N for a Few Selected Values K and $L = K/2$, Where K is a Power of 2

K	L	$N_{\max}$
16	8	5
32	16	10
64	32	21
128	64	42
256	128	85

Note. The value $N_{\max}$ is the maximum value of N for which $K \geq 3N + 2$ and $L \geq (3N + 1)/2$. For values of N not exceeding $N_{\max}$ the coefficients J_{mn} with $-N \leq m \leq N$ and $|m| \leq n \leq N$ of the Jacobian can be calculated exactly by the summation (20).

$(2\pi/K)$ and equal weights $F^K = (2\pi/K)$ if $K \geq 3N + 1$. The integral over r can be replaced by a summation over L Gauss–Legendre points r_j^L with corresponding weights G_j^L if $L \geq (3N + 1)/2$ (see Krylov [7]). So, the projection on TN of the Jacobian of a streamfunction field $\tilde{\chi}$ and a vorticity field $\hat{\eta}$, of which the first is an element of UN and the second is an element of TN, can be calculated exactly by summation if $K \geq 3N + 1$ and $L \geq (3N + 1)/2$. More explicitly,

$$J_{mn} = \frac{n+1}{\pi} \sum_{i=1}^{K} \sum_{j=1}^{L} F^K G_j^L r_j^L Y_{mn}^*(r_j^L, \theta_i^K) \times J(\tilde{\chi}(r_j^L, \theta_i^K), \hat{\eta}(r_j^L, \theta_i^K)). \tag{20}$$

The values of Y_{mn} and its derivatives with respect to r in the points (r_j^L, θ_i^K) can be calculated using recurrency relations for Y_{mn}. These relations are given in Appendix A. For the summations over i a fast Fourier transform routine is used. This routine is the most efficient if K is a power of 2. In Table II we show a few values of K and $L = K/2$ that are powers of 2 and the maximum value of N, called $N_{\max}$, for which $K \geq 3N + 1$ and $L \geq (3N + 1)/2$. Note that the functions W_{mn} in the truncation $T5$ are given in Table I. The basis functions Y_{mn} of this truncation are represented graphically in Fig. 2.

2.3. *Time Evolution and Conserved Quantities*

We are now in the position to formulate the spectral model of (1). It can be written as

$$\frac{\partial \hat{q}}{\partial t} + \mathscr{P}_N J(\tilde{\psi}, \hat{q}) + \tau\hat{\phi} + \kappa\hat{\zeta} = 0, \tag{21}$$

where we recall that $\mathscr{P}_N$ is the operator that projects on the space TN. Written out in terms of coefficients this equation reads

$$\frac{dq_{mn}}{dt} + J_{mn} + \tau\phi_{mn} + \kappa\zeta_{mn} = 0, \tag{22}$$

where the coefficients of q_{mn} and ζ_{mn} are related by $q_{mn} = \zeta_{mn} + f_{mn}$ and those of ζ_{mn} and ψ_{mn} by Eq. (17). The coefficients of the Jacobian are calculated using (20). We repeat that, due to the fact that all fields are real, we only need to keep track of the real and imaginary parts of the coefficients for nonnegative m. Following Canuto *et al.* [3] we call (21) or (22) semidiscrete, as the time dependence is still continuous. In this paper we will use a fourth-order Runge–Kutta discretization to step the system forward in time. Assuming that there is no explicit time dependence in our system the Runge–Kutta scheme can be written as follows. First we write (21) as

$$\frac{\partial \hat{q}}{\partial t} = \hat{T}(\hat{q}), \tag{23}$$

where

$$\hat{T}(\hat{q}) = -\mathscr{P}_N J(\tilde{\psi}, \hat{q}) - \tau\hat{\phi} - \kappa\hat{\zeta}. \tag{24}$$

If $\hat{q}^i$ and $\tilde{\psi}^i$ are the approximations of $\hat{q}$ and $\tilde{\psi}$ at time $t_i = i\,\Delta t$, then according to the fourth-order Runge–Kutta scheme we have

$$\hat{q}^{i+1} = \hat{q}^i + \tfrac{1}{6}\hat{F}_1 + \tfrac{1}{3}\hat{F}_2 + \tfrac{1}{3}\hat{F}_3 + \tfrac{1}{6}\hat{F}_4, \tag{25}$$

where

$$\hat{F}_1 = \Delta t \hat{T}(\hat{q}^i) \tag{26a}$$

$$\hat{F}_2 = \Delta t \hat{T}(\hat{q}^i + \tfrac{1}{2}\hat{F}_1) \tag{26b}$$

$$\hat{F}_3 = \Delta t \hat{T}(\hat{q}^i + \tfrac{1}{2}\hat{F}_2) \tag{26c}$$

$$\hat{F}_4 = \Delta t \hat{T}(\hat{q}^i + \hat{F}_3). \tag{26d}$$

For more information on time-stepping schemes we refer to Canuto *et al.* [3] and Press *et al.* [10].

Important global quantities of the system are the absolute enstrophy, circulation, angular momentum, and energy. Using that $\mathbf{v} = \mathbf{k} \times \nabla\psi$ and partial integration in the integration over r, these quantities can be written in terms of the inner product (6)

$$Q_N = \frac{\pi}{2}\langle \hat{q}, \hat{q} \rangle, \tag{27}$$

$$C_N = \pi\langle 1, \hat{\zeta} \rangle, \tag{28}$$

$$A_N = -2\pi\langle\tilde{\psi}, 1\rangle, \tag{29}$$

$$E_N = -\frac{\pi}{2}\langle\tilde{\psi}, \hat{\zeta}\rangle, \tag{30}$$

where we note that the inner product (6) can be written in terms of the coefficients ξ_{1mn} and ξ_{2mn} of ξ_1 and ξ_2,

$$\langle\xi_1, \xi_2\rangle = \sum_{m=-\infty}^{+\infty}\sum_{n=|m|}^{\infty}\frac{\xi^*_{1mn}\xi_{2mn}}{n+1}. \tag{31}$$

(For the truncated fields in the expressions above the range of m and n is, of course, finite.) In the inviscid case, i.e., the case without forcing and friction, the equations without space and time discretizations conserve absolute enstrophy, circulation as well as energy, and, if the planetary vorticity is circularly symmetric, also angular momentum. We will show that, without forcing and friction, the absolute enstrophy and the circulation are still conserved in the semidiscrete system, i.e., in the system without the Runge–Kutta time discretization. In this case also the angular momentum remains to be conserved, under the condition that the planetary vorticity is circularly symmetric. The energy, however, is not conserved. The proofs become elementary by using the following properties of the projection operator $\mathscr{P}_N$, the Laplace operator ∇^2, the Jacobian J, and the operator $\partial/\partial\theta$. First, the projection operator $\mathscr{P}_N$ is self-adjoint

$$\langle\xi_1, \mathscr{P}_N\xi_2\rangle = \langle\mathscr{P}_N\xi_1, \xi_2\rangle. \tag{32}$$

This can be proven from its definition and Eqs. (8) and (9). Also, the Laplace operator is self-adjoint if ξ_1 and ξ_2 are both zero at the boundary $r = 1$

$$\langle\xi_1, \nabla^2\xi_2\rangle = \langle\nabla^2\xi_1, \xi_2\rangle. \tag{33}$$

This can be proven by partial integration. By the same method it can be proven that the Jacobian satisfies

$$\langle\xi_1, J(\xi_2, \xi_3)\rangle = \langle J(\xi_1, \xi_2^*), \xi_3) \tag{34}$$

if either ξ_1 or ξ_2 is zero at $r = 1$. Finally, the derivative operator $\partial/\partial\theta$ satisfies

$$\left\langle\xi_1, \frac{\partial\xi_2}{\partial\theta}\right\rangle = -\left\langle\frac{\partial\xi_1}{\partial\theta}, \xi_2\right\rangle. \tag{35}$$

After these preliminaries we can write for the rate of change of absolute enstrophy in the semidiscrete system

$$\begin{aligned}\frac{dQ_N}{dt} &= \pi\left\langle\hat{q}, \frac{\partial\hat{q}}{\partial t}\right\rangle = \pi\langle\hat{q}, \mathscr{P}_N J(\hat{q}, \tilde{\psi})\rangle = \pi\langle\mathscr{P}_N\hat{q}, J(\hat{q}, \tilde{\psi})\rangle \\ &= \pi\langle\hat{q}, J(\hat{q}, \tilde{\psi})\rangle = \pi\langle J(\hat{q}, \hat{q}), \tilde{\psi})\rangle = 0,\end{aligned} \tag{36}$$

which proves that absolute enstrophy is still conserved. For the rate of change of circulation we have

$$\begin{aligned}\frac{dC_N}{dt} &= \pi\left\langle 1, \frac{\partial\hat{\zeta}}{\partial t}\right\rangle = \pi\langle 1, \mathscr{P}_N J(\hat{q}, \tilde{\psi})\rangle \\ &= \pi\langle 1, J(\hat{q}, \tilde{\psi})\rangle = \pi\langle J(1, \hat{q}), \tilde{\psi}\rangle = 0,\end{aligned} \tag{37}$$

which shows that also the circulation is conserved in the semidiscrete system. To prove that for a circularly symmetric planetary vorticity f also the angular momentum is conserved in the semidiscrete system we note that $1 = \nabla^2\rho$, where $\rho = \frac{1}{4}(r^2 - 1)$. This enables us to write

$$\begin{aligned}\frac{dA_N}{dt} &= -2\pi\left\langle\frac{\partial\tilde{\psi}}{\partial t}, 1\right\rangle = -2\pi\left\langle\frac{\partial\tilde{\psi}}{\partial t}, \nabla^2\rho\right\rangle = -2\pi\left\langle\frac{\partial\hat{\zeta}}{\partial t}, \rho\right\rangle \\ &= 2\pi\langle\mathscr{P}_N J(\tilde{\psi}, \hat{q}), \rho\rangle = 2\pi\langle J(\tilde{\psi}, \hat{q}), \mathscr{P}_N\rho\rangle \\ &= 2\pi\langle J(\tilde{\psi}, \hat{q}), \rho\rangle = 2\pi\langle\tilde{\psi}, J(\hat{q}, \rho)\rangle \\ &= -\pi\left\langle\tilde{\psi}, \frac{\partial\hat{q}}{\partial\theta}\right\rangle = -\pi\left\langle\tilde{\psi}, \frac{\partial\hat{\xi}}{\partial\theta}\right\rangle\zeta = 0.\end{aligned} \tag{38}$$

Note that the assumption of circular symmetry of the planetary vorticity allowed us to use that $\partial\hat{q}/\partial\theta = \partial\hat{\zeta}/\partial\theta$. The last expression in (38) is zero because

$$\left\langle\tilde{\psi}, \frac{\partial\hat{\zeta}}{\partial\theta}\right\rangle = -\left\langle\frac{\partial\tilde{\psi}}{\partial\theta}, \hat{\zeta}\right\rangle = -\left\langle\frac{\partial\hat{\zeta}}{\partial\theta}, \tilde{\psi}\right\rangle = -\left\langle\tilde{\psi}, \frac{\partial\hat{\zeta}}{\partial\theta}\right\rangle. \tag{39}$$

This proves that in the inviscid case the semidiscrete system conserves angular momentum if the planetary vorticity f is circularly symmetric. To derive an expression for the rate of change of the energy E_N we write

$$\tilde{\psi} = \tilde{\psi}_l + \tilde{\psi}_h, \tag{40}$$

where $\tilde{\psi}_l$ is that part of $\tilde{\psi}$ given by the coefficients ψ_{mn} with $n \leq N$ and $\tilde{\psi}_h$ the part given by the higher coefficients. We then have, as $\mathscr{P}_N\tilde{\psi} = \tilde{\psi}_l = \tilde{\psi} - \tilde{\psi}_h$,

$$\begin{aligned}\frac{dE_N}{dt} &= -\pi\left\langle\tilde{\psi}, \frac{\partial\hat{\zeta}}{\partial t}\right\rangle = \pi\langle\tilde{\psi}, \mathscr{P}_N J(\tilde{\psi}, \hat{q})\rangle = \pi\langle\mathscr{P}_N\tilde{\psi}, J(\tilde{\psi}, \hat{q})\rangle \\ &= \pi\langle\tilde{\psi}, J(\tilde{\psi}, \hat{q})\rangle - \pi\langle\tilde{\psi}_h, J(\tilde{\psi}, \hat{q})\rangle \\ &= \pi\langle J(\tilde{\psi}, \tilde{\psi}), \hat{q})\rangle - \pi\langle\tilde{\psi}_h, J(\tilde{\psi}, \hat{q})\rangle \\ &= -\pi\langle\tilde{\psi}_h, J(\tilde{\psi}, \hat{q})\rangle \neq 0.\end{aligned} \tag{41}$$

This shows that in the semidiscrete system energy is not conserved. However, we see that the degree of nonconservation is determined by the amplitude of $\tilde{\psi}_h$, i.e., by that part of the streamfunction $\tilde{\psi}$ that falls outside TN. As ψ is usually a rather smooth function of the spatial coordinates r and θ, we expect the streamfunction to have a steeply decaying spectrum in terms of the basis functions Y_{mn}. We therefore expect that the amplitude of $\tilde{\psi}_h$ will fall off rapidly with increasing truncation N and the degree of nonconservation of energy to decrease rapidly with increasing resolution. The two examples discussed in Subsection 2.1 of Part II will confirm this.

With the time discretization the conservation properties of Q_N, C_N, and A_N, are lost in principle, although a sufficiently small value of Δt will keep the degree of nonconservation at an acceptable level. It was shown by Walsteijn [13], however, that it is possible to modify the Runge–Kutta integration scheme in such a way that it restores the conservation of, e.g., absolute enstrophy. We will not make use of that possibility.

2.4. *Additional Boundary Conditions*

The model as discussed until now satisfies only one boundary condition:

$$[\tilde{\psi}(r, \theta)]_{r=1} = 0. \tag{42}$$

By construction, the inverse Laplacian (16b) has the property that for every $\hat{\zeta}$ in the space TN the corresponding $\tilde{\psi}$ in the space UN satisfies this condition. To check that this is true, we start with

$$\hat{\zeta}(r, \theta) = \sum_{m=-N}^{N} \sum_{n=|m|}^{N} \zeta_{mn} Y_{mn}(r, \theta) \Rightarrow \tag{43}$$

$$\tilde{\psi}(r, \theta) = \sum_{m=-N}^{N} \sum_{n=|m|}^{N} \zeta_{mn} \nabla^{-2} Y_{mn}(r, \theta) \Rightarrow \tag{44}$$

$$[\tilde{\psi}(r, \theta)]_{r=1} = \sum_{m=-N}^{N} \sum_{n=|m|}^{N} \zeta_{mn} [\nabla^{-2} Y_{mn}(r, \theta)]_{r=1}. \tag{45}$$

According to (A13) of Appendix A the values of W_{mn} at $r = 1$ are given by

$$[W_{mn}(r)]_{r=1} = 1, \tag{46}$$

from which it can be deduced, with the help of (16b),

$$[\nabla^{-2} Y_{mn}(r, \theta)]_{r=1} = 0, \tag{47}$$

so that (42) is indeed satisfied.

We now consider the case in which a viscosity term is added to our system, so that the right-hand side of (1) is $\nu \nabla^2 \zeta$. For simplicity we consider the case in which the planetary vorticity is zero and in which there is no vorticity forcing nor Ekman friction. If the boundary moves with a prescribed velocity $u_b(\theta)$, then the presence of viscosity implies that an additional boundary condition of no-slip needs to be imposed:

$$\left[\frac{\partial \tilde{\psi}(r, \theta)}{\partial r}\right]_{r=1} = u_b(\theta). \tag{48}$$

As the streamfunction is completely determined from the vorticity by (16b) or (17b), any additional boundary condition must be incorporated in terms of a constraint on the vorticity. How this works out for the no-slip boundary condition will be explained now. We first assume that $u_b(\theta)$ can be written as a finite Fourier series of complex exponentials:

$$u_b(\theta) = \sum_{m=-N}^{N} u_{bm} e^{im\theta}. \tag{49}$$

From (44) we have, on the other hand,

$$\left[\frac{\partial \tilde{\psi}(r, \theta)}{\partial r}\right]_{r=1} = \sum_{m=-N}^{N} \sum_{n=|m|}^{N} \zeta_{mn} \left[\frac{\partial \nabla^{-2} Y_{mn}(r, \theta)}{\partial r}\right]_{r=1}. \tag{50}$$

According to (A14) of Appendix A the radial derivatives of W_{mn} at $r = 1$ are given by

$$\left[\frac{dW_{mn}(r)}{dr}\right]_{r=1} = \frac{n^2 + 2n - m^2}{2}, \tag{51}$$

from which it follows, due to (16b),

$$\left[\frac{\partial \nabla^{-2} Y_{mn}(r, \theta)}{\partial r}\right]_{r=1} = \begin{cases} [2(n+1)]^{-1} e^{im\theta} & (n = |m|) \\ 0 & (n \geq |m| + 2). \end{cases} \tag{52}$$

It can therefore be deduced that

$$\left[\frac{\partial \tilde{\psi}(r, \theta)}{\partial r}\right]_{r=1} = \sum_{m=-N}^{N} \frac{\zeta_{m|m|}}{2(|m|+1)} e^{im\theta}, \tag{53}$$

so that the boundary condition (48) reduces to

$$\zeta_{m|m|} = 2(|m| + 1) u_{bm}. \tag{54}$$

So, the no-slip boundary condition fixes the coefficients $\zeta_{m|m|}$ in terms of the Fourier coefficents u_{bm} of the velocity of the boundary.

The results of the foregoing paragraph imply that the vorticity $\hat{\zeta}$ is to be written as

$$\hat{\zeta} = \hat{\zeta}_b + \hat{\zeta}_h, \tag{55}$$

where $\hat{\zeta}_b$ is the part of the vorticity field of which the coefficients $\zeta_{m|m|}$ are fixed by the boundary condition (54) with all higher coefficients zero, and $\hat{\zeta}_h$ is a vorticity field of which the coefficients ζ_{mn} with $n \geq |m| + 2$ are free but of which the coefficients $\zeta_{m|m|}$ are zero. Let us assume that we start a model integration with a vorticity field that satisfies condition (54). It is then important that the model dynamics is such that (54) remains valid, i.e., that the coefficients $\zeta_{m|m|}$ do not change in time and (55) remains a valid decompostition. When the planetary vorticity, the vorticity forcing, and the Ekman friction are zero but there is a viscosity term, the semidiscrete form of the model is given by (23) and (24) with $\hat{q}$ replaced by $\hat{\zeta}$ and $-\tau\hat{\phi} - \kappa\hat{\zeta}$ replaced by $\nu\nabla^2\hat{\zeta}$. A straightforward way of making sure that the coefficients $\zeta_{m|m|}$ do not change in time is to replace the tendency term $\hat{T}(\hat{\zeta})$ in (23) by $\mathscr{P}_N^h\hat{T}(\hat{\zeta})$, where $\mathscr{P}_N^h$ is a projection operator that projects on vorticity fields of which the coefficients $\zeta_{m|m|}$ are zero. There are, however, two problems with this approach. First, it can be shown by simple analytical examples that in this case the viscosity term causes unlimited growth, independent of the temporal or spatial discretization. Second, it suppresses the advection of vorticity to such a degree that the time evolution becomes unrealistically slow. After experimenting with several alternatives, the following procedure was chosen instead. We start by writing the tendency term as

$$\hat{T}(\hat{\zeta}) = \nabla^2\tilde{S}(\hat{\zeta}), \tag{56}$$

where

$$\tilde{S}(\hat{\zeta}) = -\nabla^{-2}\mathscr{P}_N J(\tilde{\psi}, \hat{\zeta}) + \nu\hat{\zeta}. \tag{57}$$

This is an exact expression as $\nabla^2\nabla^{-2}$ is the identity operator in the space TN. The field $\tilde{S}$, which is an element of the space UN, is now modified by requiring that the Laplacian of $\tilde{S}$ does not project on the functions $Y_{m|m|}$. The latter requirement can be written as

$$\langle Y_{m|m|}, \nabla^2\tilde{S}\rangle = 0. \tag{58}$$

To formulate it in terms of the coefficients S_{mn} of $\tilde{S}$ we note that, using (16a) in the second equality and changing the order of the summations in the third equality,

$$\begin{aligned}\nabla^2\tilde{S} &= \sum_{m=-N}^{N}\sum_{n=|m|+2}^{N+2} S_{mn}\nabla^2 Y_{mn} \\ &= \sum_{m=-N}^{N}\sum_{n=|m|+2}^{N+2} S_{mn}\left[\sum_{n'=|m|}^{n-2}(n'+1)(n-n')(n+n'+2)Y_{mn'}\right] \\ &= \sum_{m=-N}^{N}\sum_{n'=|m|}^{N}\left[\sum_{n=n'+2}^{N+2} S_{mn}(n'+1)(n-n')(n+n'+2)\right]Y_{mn'},\end{aligned} \tag{59}$$

from which we deduce that, with the help of (7),

$$\langle Y_{m|m|}, \nabla^2\tilde{S}\rangle = \sum_{n=|m|+2}^{N+2} S_{mn}(n-|m|)(n+|m|+2). \tag{60}$$

The requirement thus assumes the form

$$\sum_{n=|m|+2}^{N+2} S_{mn}(n-|m|)(n+|m|+2) = 0. \tag{61}$$

This is a linear constraint on the coefficients S_{mn} for $|m| + 2 \leq n \leq N + 2$. It can be satisfied in many ways. In practice we adjust the coefficients S_{mn} with the highest value of n (either $n = N + 2$ or $n = N + 1$, depending on the value of m) so that (61) is satisfied. The model equations for the viscous case—with the planetary vorticity, vorticity forcing, and Ekman friction assumed to be zero—are thus given by (23), (56), and (57), where $\hat{q} = \hat{\zeta}$ and $\tilde{S}$ is modified such that the coefficients S_{mn} satisfy (61).

In this last subsection we have discussed only the no-slip boundary condition, as this is a widely used and physically sensible boundary condition if the flow is viscous. Other boundary conditions can also be imposed, e.g., the condition that $\hat{\zeta}$ has a prescribed distribution at the boundary. This condition has been used in long simulations of two-dimensional weakly viscous flow. This, and other possible boundary conditions, can be incorporated in analogous ways.

Here the mathematical formulation of the spectral model ends. The numerical code for this model was written in FORTRAN on the basis of a (hemi)spheric spectral model—coded earlier by the author—for two-dimensional fluid flow on a sphere. More details of the present model and a discussion of six examples of time integrations can be found in the companion paper (Part II).

3. SUMMARY

We have developed a spectral numerical model for two-dimensional incompressible fluid flow in a circular basin. The equation governing such flow is (1), where q is the absolute vorticity—the sum of the relative vorticity ζ and the planetary vorticity f—and ψ is the streamfunction. The absolute vorticity q is advected by a two-dimensional divergence-free velocity field $\mathbf{v} = \mathbf{k} \times \nabla\psi$ as expressed by the Jacobian of ψ and q. The streamfunction is zero at the boundary, which implies that there is no normal flow at the boundary (free-slip). Apart from being advected, the absolute vorticity is forced by a vorticity forcing $-\tau\phi$, where ϕ is the spatial structure of the forcing and τ is its amplitude. The vorticity is damped by a term proportional to the relative vorticity, i.e., by $-\kappa\zeta$, which is called Ekman friction. Also considered is the case in which the right-hand side of (1) equals $\nu\nabla^2\zeta$, where ν is the viscosity. For this viscous flow we impose as an extra boundary condition that the radial derivative of ψ is equal to the velocity u_b of the boundary (no-slip).

Vorticity fields are written in terms of the functions $Y_{mn}(r, \theta)$, defined in (2) and (3), where r and θ are polar coordinates, $0 \leq r \leq 1$, $0 \leq \theta < 2\pi$, and m and n are integers, $-N \leq m \leq N$, $|m| \leq n \leq N$ with $n = |m| + 2k$, $k = 0, 1, 2, ...$ The functions $Y_{mn}(r, \theta)$ are products of the complex exponentials $e^{im\theta}$ and the functions $r^{|m|}P_k^{(0,|m|)}(s)$, where $P_k^{(\alpha,\beta)}(s)$ are Jacobi polynomials with argument $s = 2r^2 - 1$ (see Abramowitz and Stegun [1, Chap. 22]). The number N is called the truncation limit and the vector space spanned by the functions Y_{mn} is called TN. The inverse Laplacian of each function Y_{mn} in TN—such that it is zero at the boundary—turns out to be a linear combination of Y_{mn-2}, Y_{mn}, and Y_{mn+2}. As the inverse Laplacian is a linear operator, this can be used to find the streamfunction associated with any vorticity field in TN. Indeed, the streamfunction of any vorticity field in TN is the sum of the streamfunctions associated with each of its individual components and will be an element of a higher-dimensional space that we call UN. For a vorticity field that is an element of the space TN and a streamfunction field that is an element of UN, the Jacobian of the streamfunction and the vorticity as projected on the space TN can be calculated exactly by summation using at least $3N + 1$ equidistant θ values and $(3N + 1)/2$ Gaussian r values. The resulting spectral numerical scheme without time discretization and without forcing and friction is shown to conserve circulation, angular momentum (if the planetary vorticity is circularly symmetric), and absolute enstrophy, although energy is not conserved.

From the foregoing it follows that if $\psi = 0$ at $r = 1$ is the only boundary condition, the vorticity fields within the space TN are free, i.e., do not satisfy any constraints. The presence of a viscosity term, and the additional boundary condition that $\partial\psi/\partial r = u_b$ at $r = 1$, leads to a constraint on the vorticity which is treated as follows. First, we obtain explicit expressions of the radial derivative of ψ at $r = 1$ in terms of the coefficients of the vorticity. This is possible because, as just explained, the streamfunction is completely fixed by the vorticity. We then require that the radial derivative of ψ at $r = 1$ is equal to the velocity u_b of the boundary. It can be deduced that this requirement fixes the expansion coefficients $\zeta_{m|m|}$ of the vorticity in terms of the Fourier coefficients of the velocity of the boundary, as expressed by (54). In order to ensure that the dynamics does respect this condition, the tendency term $\hat{T}$ in (23) is written as the Laplacian of a field $\tilde{S}$—given by (57)—which field is then modified such that the tendency term does not project on the basis functions $Y_{m|m|}$. The condition that the tendency term does not project on the functions $Y_{m|m|}$ is given by (61) and is a linear relation in terms of the coefficients S_{mn} with $|m| + 2 \leq n \leq N + 2$. It is satisfied by adjusting the highest coefficients S_{mn} to the lower ones.

We recall that the basis functions Y_{mn} (defined in (2) and (3)) span the same linear vector space as the functions X_{mn} (defined in (4) and (5)) which, in turn, span the same linear vector space as the functions 1, x, y, x^2, xy, y^2, x^3, x^2y, xy^2, y^3, etc. The functions X_{mn} emerge naturally from the perturbation approach to the simple wind-driven ocean circulation problem studied by Verkley and Zimmerman [12]. The functions Y_{mn} can, in fact, be obtained by orthogonalizing the functions X_{mn} with respect to the inner product (6). The functions X_{mn} and Y_{mn} can be transformed into each other and this makes it possible, e.g., to derive analytic expressions for the Laplacian and its inverse in terms of Y_{mn}. It also makes clear that the functions Y_{mn} do not suffer from a "pole problem"; i.e., they are infinitely differentiable at $r = 0$.

Without this particular background (or bias) one might have proposed other basis functions, like products of Bessel functions or Chebyshev polynomials in r and complex exponentials in θ. Although Bessel functions have the advantage of giving basis functions that are eigenfunctions of the Laplacian, the calculation of the Jacobian coefficients in (20) with Gauss–Legendre points would have been impossible. With Chebyshev polynomials this would have been possible indeed—in fact the calculation could be faster because a fast Fourier transform can then also be applied in the integration over r—but the integration points to be used cluster near $r = 0$, necessitating a very short time step in the time discretization. However, other possibilities exist and a very general approach to the problem can be found in Dubiner [4]. In fact, the use of the functions Y_{mn} (as we have called them) was first suggested in this reference. Matsushima and Marcus [8] focus on a spectral method for polar coordinates and provide a wide class of basis functions that do not suffer from the problems mentioned above. The basis functions that they propose are products of complex exponentials in θ and functions called $Q_n^m(\alpha, \beta; r)$, defined by their Eq. (3). Recurrency relations, convergence properties, and the representation of different operators are being studied, and two examples are given to illustrate their use. The first of these examples concerns the eigenvalue problem that leads to Bessel functions, and it was concluded that with $\alpha = 1$ and $\beta = 1$

their functions give the best convergence properties. As it happens, for $\alpha = 1$ and $\beta = 1$, their $Q_n^m(\alpha, \beta; r)$ is identical to our $W_{mn}(r)$, as can be verified from their Eq. (3) and our Eqs. (3) and (A6).

APPENDIX A

The Jacobi Polynomials

In this appendix we discuss some basic properties of the functions $W_{mn}(r) = r^{|m|}P_k^{(0,|m|)}(s)$. We start by defining the Jacobi polynomials $P_k^{(0,m)}(s)$ in terms of hypergeometric functions. Then two recurrency relations are given by means of which the functions $W_{mn}(r)$ and their derivatives with respect to r can be calculated in a numerically stable way. After this the relationship with the functions X_{mn} is discussed. This relationship is used in the calculation of the Laplacian and its inverse in terms of the functions Y_{mn}. In the latter calculations a theorem on Pochhammer symbols is used, which is proved in Appendix B.

A.1. *Definitions*

Several equivalent definitions of Jacobi polynomials $P_k^{(\alpha,\beta)}(s)$ can be found in Chapter 22 of Abramowitz and Stegun [1] and Chapter 10 of Erdélyi [5]. In the definition of Y_{mn} we need these polynomials for $\alpha = 0$, $\beta = |m|$, and $k = (n - |m|)/2$. In the following we will delete the absolute value signs around m and assume, without loss of generality, that m is nonnegative. A convenient expression of the Jacobi polynomials is (16) from Section 10.8 of Erdélyi [5]. For $\alpha = 0$ and $\beta = m$ it follows from this expression that

$$P_k^{(0,m)}(s) = (-1)^k \binom{m+k}{k} \times F\left(-k, m+1+k; m+1; \frac{1+s}{2}\right), \tag{A1}$$

where F is the hypergeometric function, defined in (15.1.1) of Abramowitz and Stegun [1]:

$$F(a, b; c; z) \equiv \sum_{k'=0}^{\infty} \frac{(a)_{k'}(b)_{k'}}{(c)_{k'}} \frac{z^{k'}}{k'!}. \tag{A2}$$

Here ( ) is the binomial coefficient and $(z)_i$ is the Pochhammer symbol, defined by $(z)_i = \Gamma(z+i)/\Gamma(z)$ or $(z)_0 = 1$ and $(z)_i = z(z+1)\cdots(z+i-1)$ for $i = 1, 2, \ldots$. The symbol Γ denotes the Gamma function, whose properties are given in Chapter 6 of Abramowitz and Stegun [1]. It can be verified easily that

$$\binom{m+k}{k} = \frac{(m+1)_k}{k!}, \tag{A3}$$

$$(-k)_{k'} = \begin{cases} (-1)^{k'} k![(k-k')!]^{-1}, & k' \le k \\ 0, & k' > k, \end{cases} \tag{A4}$$

$$(m+1)_k(m+1+k)_{k'} = (m+1)_{k'}(m+1+k')_k. \tag{A5}$$

To prove the latter equation we used the definition of the Pochhammer symbol in terms of Γ functions. Substituting (A2) into (A1) and using the expressions above we obtain

$$P_k^{(0,m)}(s) = (-1)^k \sum_{k'=0}^{k} (-1)^{k'} \frac{(m+1+k')_k}{(k-k')!k'!} r^{2k'}. \tag{A6}$$

We therefore see that $P_k^{(0,m)}(s)$ is a polynomial in r with degree $2k$. The fact that the functions Y_{mn} with different values of n are orthogonal can be demonstrated by noticing that for the inner product of $Y_{mn'}$ and Y_{mn}, where $n = m + 2k$ and $n' = m + 2k'$, we have

$$\langle Y_{mn'}, Y_{mn} \rangle = (1/2)^{m+1} \int_{-1}^{1} ds(1+s)^m P_{k'}^{(0,m)}(s) P_k^{(0,m)}(s). \tag{A7}$$

According to (22.1.1), (22.1.2), and (22.2.1) of Abramowitz and Stegun [1] the right-hand side of this expression is equal to $(2k+m+1)^{-1}\delta_{kk'} = (n+1)^{-1}\delta_{nn'}$. We note that the squared norm of the functions Y_{mn} is equal to the squared norm of the functions X_{mn}.

A.2. *Recurrency Relations*

The starting point for the recurrency relations that we need are (22.7.1) and (22.8.1) of Abramowitz and Stegun [1]. For the functions $P_k^{(0,m)}(s)$ these relationships read

$$\begin{aligned} 2(k+1)(m+1+k)(m+2k)P_{k+1}^{(0,m)}(s) \\ = [-(m+1+2k)m^2 \\ + (m+2k)(m+1+2k)(m+2+2k)s]P_k^{(0,m)}(s) \\ - 2k(m+k)(m+2+2k)P_{k-1}^{(0,m)}(s) \end{aligned} \tag{A8}$$

and

$$\begin{aligned} (m+2k)(1-s^2)\frac{d}{ds}P_k^{(0,m)}(s) \\ = -k[m + (m+2k)s]P_k^{(0,m)}(s) + 2k(m+k)P_{k-1}^{(0,m)}(s). \end{aligned} \tag{A9}$$

From the relations above it can then be readily verified that for the functions $W_{mn}(r)$ we have

$$W_{mn+2}(r) = \left[\frac{-(m+1+2k)m^2}{2(k+1)(m+1+k)(m+2k)} + \frac{(m+1+2k)(m+2+2k)}{2(k+1)(m+1+k)}s\right]W_{mn}(r) - \frac{k(m+k)(m+2+2k)}{(k+1)(m+1+k)(m+2k)}W_{mn-2}(r) \quad \text{(A10)}$$

and

$$r\frac{d}{dr}W_{mn}(r) = \left(m - \frac{2k[m+(m+2k)s]}{(m+2k)(1-s)}\right)W_{mn}(r) + \frac{4k(m+k)}{(m+2k)(1-s)}W_{mn-2}(r). \quad \text{(A11)}$$

It follows from (A6) that

$$W_m(r) = r^m, \quad \text{(A12a)}$$

$$W_{m+2}(r) = r^m[-(m+1)+(m+2)r^2]. \quad \text{(A12b)}$$

For any value of m we can then use (A10) to calculate all the functions $W_{mn}(r)$ with $n = m, m+2, m+4$, etc., in any point r. Subsequently, relation (A11) can be used to calculate the derivatives with respect to r of these functions. From (A10) and (A12) it can be proven that we have for $W_{mn}(r)$ at $r = 1$

$$[W_{mn}(r)]_{r=1} = 1. \quad \text{(A13)}$$

The proof is by induction. From (A12) it follows that (A13) is true for m and $m+2$ and (A10) can be used to prove that it is true for all higher values of n. In the same way, although with considerable more algebra, it can be proven from (A11) and (A12) that

$$\left[\frac{dW_{mn}(r)}{dr}\right]_{r=1} = \frac{n^2+2n-m^2}{2}. \quad \text{(A14)}$$

Again, the proof is by induction, using (A12) in the starting step and (A11) in the induction step.

A.3. *Relations between X_{mn} and Y_{mn}*

From expression (A6) it is seen immediately that Y_{mn} can be expressed in terms of X_{mn}. We have, writing $n' = m + 2k'$,

$$Y_{mn} = (-1)^k \sum_{k'=0}^{k} (-1)^{k'} \frac{(m+1+k')_k}{(k-k')!k'!} X_{mn'}. \quad \text{(A15)}$$

It is also possible to express X_{mn} in terms of Y_{mn}. According to (8) and (9) and the fact that both X_{mn} and Y_{mn} are polynomials in r of degree $2k$ we can write

$$X_{mn} = \sum_{k'=0}^{k} (m+1+2k')\langle Y_{mn'}, X_{mn}\rangle Y_{mn'}. \quad \text{(A16)}$$

For the inner product we have

$$\langle Y_{mn'}, X_{mn}\rangle = (1/2)^{m+k+1}\int_{-1}^{1} ds(1+s)^{m+k}P_{k'}^{(0,m)}(s) = \frac{\Gamma(k+1)\Gamma(m+1+k)}{\Gamma(k-k'+1)\Gamma(m+2+k+k')}, \quad \text{(A17)}$$

where in the latter equality we used Eq. (1) of Section 16.4 from Erdélyi [5].[1] Now, using elementary properties of the Γ function, it can be shown that

$$\frac{\Gamma(k+1)}{\Gamma(k-k'+1)} = (k-k'+1)_{k'}, \quad \text{(A18)}$$

$$\frac{\Gamma(m+1+k)}{\Gamma(m+2+k+k')} = \frac{1}{(m+1+k+k')(m+1+k)_{k'}}, \quad \text{(A19)}$$

so that

$$\langle Y_{mn'}, X\mathrm{mn}\rangle = \frac{(k-k'+1)_{k'}}{(m+1+k+k')(m+1+k)_{k'}}. \quad \text{(A20)}$$

Substituting this in (A16) gives

$$X_{mn} = \sum_{k'=0}^{k} \frac{(m+1+2k')}{(m+1+k+k')}\frac{(k-k'+1)_{k'}}{(m+1+k)_{k'}} Y_{mn'}, \quad \text{(A21)}$$

which is the desired result.

A.4. *The Laplacian ∇^2 and Its Inverse ∇^{-2}*

It can be verified easily that

$$\nabla^2 X_{mn} = 4k(m+k)X_{mn-2}. \quad \text{(A22)}$$

We apply the Laplacian to (A15), use the equation above, and then use (A21) with k and k' replaced by k' and k''. We next change the order of the summations over k and k' and interchange primed and doubly primed variables to obtain

$$\nabla^2 Y_{mn} = \sum_{k'=0}^{k-1} L_{kk'} Y_{mn'}, \quad \text{(A23)}$$

[1] This expression should contain an extra $n!$ in the denominator, according to Eq. (11) on page 583 of Section 2.22.2 of Prudnikov *et al.* [11].

with

$$L_{kk'} = 4(m + 1 + 2k')(-1)^k \sum_{k''=k'+1}^{k} (-1)^{k''} \times \frac{(m + 1 + k'')_k}{(k - k'')!k''!} k'' \frac{(m + k'')(k'' - k')_{k'}}{(m + k' + k'')(m + k'')_{k'}}. \quad \text{(A24)}$$

To simplify this expression we use

$$\frac{(m + k'')}{(m + k' + k'')} \frac{1}{(m + k'')_{k'}} = \frac{1}{(m + 1 + k'')_{k'}}, \quad \text{(A25)}$$

$$\frac{(m + 1 + k'')_k}{(m + 1 + k'')_{k'}} = (m + 1 + k' + k'')_{k-k'}, \quad \text{(A26)}$$

$$\frac{k''(k'' - k')_{k'}}{k''!} = \frac{1}{(k'' - k' - 1)!}, \quad \text{(A27)}$$

so that

$$L_{kk'} = 4(m + 1 + 2k')(-1)^k \sum_{k''=k'+1}^{k} (-1)^{k''} \times \frac{(m + 1 + k' + k'')_{k-k'}}{(k'' - k' - 1)!(k - k'')!}. \quad \text{(A28)}$$

Introducing the new variable $l = k'' - k' - 1$ we can rewrite this expression as

$$L_{kk'} = \frac{4(-1)^{k+k'+1}}{(k - k' - 1)!} (m + 1 + 2k') \sum_{l=0}^{k-k'-1} (-1)^l \times \binom{k - k' - 1}{l} (m + 2 + 2k' + l)_{k-k'}. \quad \text{(A29)}$$

If we now use Theorem (B1), proved in Appendix B, for $i = k - k' - 1$, $k = 1$, and $z = m + 2 + 2k'$ we can write

$$\sum_{l=0}^{k-k'-1} (-1)^l \binom{k - k' - 1}{l} (m + 2 + 2k' + l)_{k-k'} = (-1)^{k-k'-1}(2)_{k-k'-1}(m + 1 + k + k')_1. \quad \text{(A30)}$$

This gives

$$L_{kk'} = 4(m + 1 + 2k')(k - k')(m + 1 + k + k') = (n' + 1)(n - n')(n + n' + 2), \quad \text{(A31)}$$

where we used that $n = m + 2k$ and $n' = m + 2k'$. Substituting this expression into (A23), summing over n' instead of k', we get

$$\nabla^2 Y_{mn} = \sum_{n'=m}^{n-2} (n' + 1)(n - n')(n + n' + 2)Y_{mn'}. \quad \text{(A32)}$$

This is identical to (16a). Equation (17a) can be derived by starting from expression (13) of a streamfunction $\tilde{\chi}$. We apply the result above, change the order of the summations, and interchange primed and unprimed variables after which (17a) follows immediately.

It can also be checked quite straightforwardly that

$$\nabla^{-2} X_{mn} = \frac{1}{4(k + 1)(m + k + 1)} (X_{mn+2} - X_{mm}). \quad \text{(A33)}$$

Note that the term proportional to X_{mm} has a zero Laplacian and is used to satisfy the boundary condition $[\nabla^{-2} X_{mn}(r, \theta)]_{r=1} = 0$. In the same way as we obtained (A23) and (A24) we can derive, using the expression above in combination with (A15) and (A21),

$$\nabla^{-2} Y_{mn} = \sum_{k'=0}^{k+1} L^{-1}_{kk'} Y_{mn'}, \quad \text{(A34)}$$

with

$$L^{-1}_{k0} = 1/4(-1)^k \sum_{k''=0}^{k} (-1)^{k''} \times \frac{(m + 1 + k'')_k}{(k - k'')!k''!(k'' + 1)(m + 1 + k'')} \times \left(\frac{(m + 1)}{(m + 2 + k'')} - 1\right), \quad \text{(A35a)}$$

$$L^{-1}_{kk'} = 1/4(-1)^k(m + 1 + 2k') \sum_{k''=k'-1}^{k} (-1)^{k''} \times \frac{(m + 1 + k'')_k}{(k - k'')!k''!(k'' + 1)(m + 1 + k'')} \times \frac{(k'' - k' + 2)_{k'}}{(m + 2 + k' + k'')(m + 2 + k'')_{k'}}, \quad \text{(A35b)}$$

where in the last expression $k' \geq 1$. Note that we again changed the order of the summations over k' and k'' and then interchanged primed and doubly primed variables. Using that

$$\frac{(m + 1)}{(m + 2 + k'')} - 1 = -\frac{(k'' + 1)}{(m + 2 + k'')} \quad \text{(A36)}$$

we can write for L^{-1}_{k0}

$$L^{-1}_{k0} = 1/4(-1)^{k+1}\sum_{k''=0}^{k}(-1)^{k''} \times \frac{(m+1+k'')_k}{(k-k'')!k''!(m''+1+k'')(m+2+k'')}. \tag{A37}$$

It can be checked readily from this expression that

$$L^{-1}_{0,0} = \frac{-1}{4(m+1)(m+2)}, \tag{A38a}$$

$$L^{-1}_{1,0} = \frac{1}{4(m+2)(m+3)}. \tag{A38b}$$

For $k \geq 2$ we use that

$$\frac{(m+1+k'')_k}{(m+1+k'')(m+2+k'')} = (m+3+k'')_{k-2} \tag{A39}$$

to find

$$L^{-1}_{k0} = 1/4(-1)^{k+1}1/k!\sum_{k''=0}^{k}(-1)^{k''} \times \binom{k}{k''}(m+3+k'')_{k-2} = 0, \tag{A40}$$

where the last equality follows from Theorem (B1) for $i = k$, $k = -2$, and $z = m + 3$. So, $L^{-1}_{k0} = 0$ if $k \geq 2$. Expression (A35b) can be simplified by using

$$(m+1+k'')(m+2+k'')_{k'}(m+2+k'+k'') = (m+1+k'')_{k'+2}, \tag{A41}$$

$$\frac{(k''-k'+2)_{k'}}{k''!(k''+1)} = \frac{1}{(k''-k'+1)!}, \tag{A42}$$

which gives

$$L^{-1}_{kk'} = 1/4(-1)^k(m+1+2k')\sum_{k''=k'-1}^{k}(-1)^{k''} \times \frac{1}{(k-k'')!(k''-k'+1)!}\frac{(m+1+k'')_k}{(m+1+k'')_{k'+2}}. \tag{A43}$$

Using that for $k' \geq k - 1$, we have

$$\frac{(m+1+k'')_k}{(m+1+k'')_{k'+2}} = \frac{1}{(m+1+k+k'')_{k'-k+2}}, \tag{A44}$$

we can write in this case

$$L^{-1}_{kk'} = 1/4(-1)^k(m+1+2k')\sum_{k''=k'-1}^{k}(-1)^{k''} \times \frac{1}{(k-k'')!(k''-k'+1)!}\frac{1}{(m+1+k+k'')_{k'-k+2}}. \tag{A45}$$

It can be checked from this expression that

$$L^{-1}_{kk+1} = \frac{1}{4(m+1+2k)(m+2+2k)}, \tag{A46a}$$

$$L^{-1}_{kk} = \frac{-1}{2(m+2k)(m+2+2k)}, \tag{A46b}$$

$$L^{-1}_{kk-1} = \frac{1}{4(m+2k)(m+1+2k)}. \tag{A46c}$$

Consider next the case that $k' \leq k - 2$. Using that, for $k' \leq k - 2$,

$$\frac{(m+1+k'')_k}{(m+1+k'')_{k'+2}} = (m+3+k'+k'')_{k-k'-2}, \tag{A47}$$

we can write

$$L^{-1}_{kk'} = 1/4(-1)^k(m+1+2k')\sum_{k''=k'-1}^{k}(-1)^{k''} \times \frac{1}{(k-k'')!(k''-k'+1)!}(m+3+k'+k'')_{k-k'-2}. \tag{A48}$$

Introducing the new variable $l = k'' - k' + 1$ we can rewrite this expression as

$$L^{-1}_{kk'} = \frac{(-1)^{k+k'-1}}{4(k-k'+1)!}(m+1+2k')\sum_{l=0}^{k-k'+1}(-1)^l \times \binom{k-k'+1}{l}(m+2+2k'+l)_{k-k'-2} = 0, \tag{A49}$$

where in the last equality we used (B1) with $i = k - k' + 1$, $k = -3$, and $z = m + 2 + 2k'$. This implies that $L_{kk'} = 0$ if $k' \leq k - 2$. Substituting the resulting expressions for $L^{-1}_{kk'}$ into (A34) we finally get

$$\nabla^{-2}Y_{mn} = \begin{cases} -[4(n+1)(n+2)]^{-1}Y_{mn} + [4(n+1)(n+2)]^{-1}Y_{mn+2} & (n = m) \\ [4n(n+1)]^{-1}Y_{mn-2} - [2n(n+2)]^{-1}Y_{mn} + [4(n+1)(n+2)]^{-1}Y_{mn+2} & (n \geq m+2). \end{cases} \tag{A50}$$

This expression is identical to (16b). In the same manner as described above we can derive (17b).

APPENDIX B

A Theorem on Pochhammer Symbols

The theorem we have used on several occasions in Appendix A is

$$\sum_{j=0}^{i} (-1)^j \binom{i}{j} (z+j)_{i+k} = \begin{cases} (-1)^i (k+1)_i (z+i)_k, & k \geq 0 \\ 0, & k < 0, \end{cases} \tag{B1}$$

where i, j, and k are integers with i, j, and $i + k$ larger than or equal to 0 and z is any complex number. It can be proved by induction in terms of i. For $i = 0$ the statement is clearly true. We will assume it to be true for any positive value of i and then prove it to be true for $i + 1$. Because

$$(z+j)_{i+1+k} = (z+i+j+k)(z+j)_{i+k}, \tag{B2}$$

we have

$$\sum_{j=0}^{i+1} (-1)^j \binom{i+1}{j} (z+j)_{i+1+k} = \sum_{j=1}^{i+1} (-1)^j \binom{i+1}{j} j(z+j)_{i+k} + (z+i+k)\sum_{j=0}^{i+1} (-1)^j \binom{i+1}{j} (z+j)_{i+k}. \tag{B3}$$

Now, using that

$$\binom{i+1}{j} j = (i+1)\binom{i}{j-1}, \tag{B4}$$

and (see Abramowitz and Stegun [1, Chap. 3, Eq. (3.1.4)])

$$\binom{i+1}{j} = \binom{i}{j} + \binom{i}{j-1}, \tag{B5}$$

we can write

$$\begin{aligned} \sum_{j=0}^{i+1} (-1)^j \binom{i+1}{j} (z+j)_{i+1+k} &= (i+1)\sum_{j=1}^{i+1} (-1)^j \binom{i}{j-1} (z+j)_{i+k} \\ &\quad + (z+i+k)\sum_{j=0}^{i} (-1)^j \binom{i}{j} (z+j)_{i+k} \\ &\quad + (z+i+k)\sum_{j=1}^{i+1} (-1)^j \binom{i}{j-1} (z+j)_{i+k} \end{aligned} \tag{B6}$$

$$\begin{aligned} &= (z+i+k)\sum_{j=0}^{i} (-1)^j \binom{i}{j} (z+j)_{i+k} \\ &\quad - (z+2i+k+1)\sum_{j=0}^{i} (-1)^j \binom{i}{j} (z+1+j)_{i+k}, \end{aligned} \tag{B7}$$

where we added the first and the third term on the right-hand side of the first equality, and then changed the summation over j in the resulting term by a summation over $i = j - 1$ and changed back to the original summation index j. In this stage we can use the induction assumption to obtain

$$\begin{aligned} \sum_{j=0}^{i+1} (-1)^j \binom{i+1}{j} (z+j)_{i+1+k} &= (-1)^i (k+1)_i [(z+i+k)(z+i)_k \\ &\quad - (z+2i+k+1)(z+1+i)_k]. \end{aligned} \tag{B8}$$

This can be simplified by using

$$(z+i+k)(z+i)_k = (z+i)(z+1+i)_k \tag{B9}$$

to obtain

$$\sum_{j=0}^{i+1} (-1)^j \binom{i+1}{j} (z+j)_{i+1+k}$$
$$= (-1)^i (k+1)_i (-1)(k+1+i)(z+1+i)_k \quad \text{(B10)}$$
$$= (-1)^{i+1} (k+1)_{i+1} (z+i+1)_k.$$

This proves the theorem.

ACKNOWLEDGMENTS

I thank Dr. H. Clercx for reading the first draft of the manuscript. Drs. R. Pasmanter and H. Brands are thanked for inspiring me to go beyond the free-slip boundary condition. I am grateful to the reviewers for their constructive criticism.

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